On the Propagation of Flood Waves

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Synopsis

In natural rivers, the forms of the channels,—the bed slopes, the breadth, the forms of cross sections, etc.—are all very irregular and incessantly changing. It is impossible to grasp them definitely. Yet the flow in rivers is steady and nearly uniform in the broad means. The disturbances on the flow caused by these irregularities damp away within a few kilometres and have certain limited dimensions and durations. The stochastic character of the collective of these elementary disturbances causes a large scale longitudinal mixing. The order of magnitude of the diffusion coefficient may be estimated to be $10^6 \sim 10^8$ c. g. s. according to the scale of a river. Introducing the effect of longitudinal diffusion caused by the mixing into the equation of continuity and assuming the mean flow taken over a suitable range to be steady and uniform, the differential equation of flood waves was derived. It is an equation of diffusion containing a term of advection. As the equation is non-linear, an approximate method of solution was discussed and solutions were obtained under several conditions. They well explain the properties of flood waves. The approximate equation of flood waves is linear, a flood of any form is, therefore, supposed to be composed of many elementary flood waves of simple character,— unit graph, or unit flood. A method of computing the unit graph was described and some numerical examples were shown. In the last, some of the results of observations made on an artificial unit flood in the Yedo River were compared with the theoretical computations. Their agreement is excellent.

1. Longitudinal Mixing in Rivers

(1) Nature of the flow in rivers

The flow in rivers is usually treated as one dimensional flow. The equation of motion is approximately given by

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}=\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right)-g\frac{\partial H}{\partial x}+gi\,, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where
g acceleration of gravity
u velocity of flow
H depth of the river
i slope of the river bed
η turbulent coefficient
t time
x coordinate axis taken along the river and positive downstream
z coordinate axis taken vertically and positive downward,

with the boundary conditions:

$$\left.\begin{array}{lll} \text{at} & x=0, & u(0,\ t)=u_0\,, \\ & z=0, & \eta\dfrac{\partial u}{\partial z}=-T\,, \\ & z=H, & \eta\dfrac{\partial u}{\partial z}=-\gamma u_b^2\,, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where u_b is the bottom velocity and T is the wind stress, if any, and the initial condition:

$$\text{at} \quad t=0, \qquad u(x,\ 0)=u_1\,.$$

Integrating the eq. (1) with respect to z from the surface to the bottom and dividing by H, we get

$$\frac{\partial U}{\partial t}+\frac{\alpha}{2}\frac{\partial U^2}{\partial x}=-\beta U^2+g\left(i-\frac{\partial H}{\partial x}\right)+\frac{T}{H}\,, \quad \ldots\ldots\ldots\ldots (3)$$

where

$$U=\frac{1}{H}\int_0^H u\,dz, \quad \alpha U^2=\frac{1}{H}\int_0^H u^2 dz\,,$$

and

$$\beta U^2=\frac{\gamma u_b^2}{H}\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

So far as the vertical profile of the velocity distribution is variable, the factors α, β change with time and position. In the river hydraulics they are, however, treated usually as certain constants which means the flow is practically steady and nearly uniform in ordinary channels. Permitting this assumption to hold, we divide the mean velocity U into two parts such as

$$U=U_0+\delta U, \quad U_0>\delta U, \quad \overline{\delta U}=0\,,$$

where U_0 is the mean steady and uniform velocity and δU is the fluctuating velocity. They satisfy the differential equations

$$-\beta U_0^2+g\left(i_0-\frac{\partial H_0}{\partial x}\right)=0\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and

$$\frac{\partial \delta U}{\partial t}+\alpha U_0\frac{\partial \delta U}{\partial x}=-2\beta U_0\delta U+g\left(i_1-\frac{\partial H_1}{\partial x}\right)+\frac{T}{H}\,, \quad \ldots\ldots\ldots (6)$$

respectively, where

$$i=i_0+i_1, \qquad H=H_0+H_1\,.$$

It will be reasonably assumed that the irregularities i_1, H_1 and T are composed of steady part and non-steady part. We assume

$$\left.\begin{array}{lll} \text{steady part of} & g\left(i_1-\dfrac{\partial H_1}{\partial x}\right)+\dfrac{T}{H}=0, & x\leqslant 0\,, \\ & =D_1(x)\neq 0, & x>0, \\ \text{non-steady part of} & g\left(i_1-\dfrac{\partial H_1}{\partial x}\right)+\dfrac{T}{H}=0, & x,\ t \quad 0, \\ & =D_2(x,\ t)\neq 0, & x,\ t>0. \end{array}\right\} \quad (7)$$

Then the solution of the eq. (6) under the conditions

$$\delta U(x,\ 0)=0 \quad \text{and} \quad \delta U(0,\ t)=0\,,$$

is given by

$$\delta U=\frac{1}{\alpha U_0}\int_{x-\alpha U_0 t}^{x} D_1(\xi)e^{-\frac{2\beta}{\alpha}(x-\xi)}d\xi+\int_0^t D_2\{x-\alpha U_0(t-\tau),\ \tau\}e^{-2\beta U_0(t-\tau)}d\tau \quad \ldots (8)$$

When the irregularities i_1, H_1 and T are localized at ξ_1 and τ_1, their effects upon δU are given by

$$\delta U=\frac{D(\xi_1)}{\alpha U_0}e^{-\frac{2\beta}{\alpha}(x-\xi_1)}+D_2(\xi_1,\ \tau_1)e^{-2\beta U_0(t-\tau_1)} \quad \ldots\ldots\ldots\ldots (9)$$

Since it has damping factors exp $\{-2\beta U_0(t-\tau)\}$ and exp $\left\{-\dfrac{2\beta}{\alpha}\right.$

$\times (x-\xi)\Big\}$ for t and x respectively, any elementary disturbance damps away within finite time interval and distance, in other words, it has finite dimension and duration. In rivers of a few metres depth with the bed of sand or gravel, the elementary disturbance will damp within a few kilometres and several ten minutes.

(2) Statistical effect of elementary disturbances.

As is shown in the formula (8), the effect of the irregularities upon δU is the superposition of those of the elementary irregularities localized at any time and position. In actual rivers of movable bed, i_1, H_1 and T fluctuate very irregularly with respect to t and x and we can not definitely grasp the true picture of them not only practically, but also in principle, because some stochastic process underlies the phenomena. It will be, therefore, reasonably inferred that the collective of δU constitutes a sort of turbulence, the elements of which are given by (9). The most important statistical effect of turbulence is the phenomena of diffusion. By the analogous reasoning with the ordinary eddy diffusion, the diffusion coefficient in our case will be given by the expression

$$\overline{\delta U \cdot R}\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where R is the dimension of an elementary disturbance. If we take δU to be several ten centimetres per second and R a few kilometres, then the diffusion coefficient will assume the value of the order of $10^6 \sim 10^7$ c. g. s.. In large rivers such as the Yangtzekiang, the Mississippi it will be of the order of 10^8 c. g. s.. By this process of longitudinal diffusion, any physical quantity θ of conservative character, if its mean value is taken over a few kilometres or over several ten minutes, will be transported downstream per unit time through unit cross section of a river by the amount

$$-\overline{\delta U \cdot R}\frac{\overline{\partial \theta}}{\partial x}\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

The diffusion phenomena described here constitute the basis of the following discussions.

2. Differential Equation of Flood Waves

(1) Formulation of differential equation

We assume, for the simplicity sake, that in the mean the channel of a river is uniform and has a rectangular cross section. Further, we assume the mean motion is uniform and steady and the collective of fluctuating disturbances due to all irregularities constitute a sort of longitudinal turbulence resulting in a phenomena of horizontal diffusion. A vertical column of water is then transported downstream not only by the mean stream, but also by the action of diffusion. The equation of motion now becomes

$$U=c\sqrt{H\left(i-\frac{\partial H}{\partial x}\right)}\,, \qquad \text{(12)}$$

where U denotes the mean velocity and

$$c^2=\frac{g}{\beta H}\,,$$

and the equation of continuity is given by

$$\frac{\partial H}{\partial t}=-\frac{\partial Q}{\partial x}+\eta\frac{\partial^2 H}{\partial x^2}\,, \qquad \text{(13)}$$

where

$$Q=U\cdot H, \qquad \eta=\overline{\delta U\cdot R}\,.$$

Putting the eq. (12) into the eq. (13), we get the equation of flood waves

$$\frac{\partial H}{\partial t}+\frac{3U}{2}\frac{\partial H}{\partial x}=\mu\frac{\partial^2 H}{\partial x^2}\,, \qquad \text{(14)}$$

where

$$\mu=\frac{HU}{2\left(i-\frac{\partial H}{\partial x}\right)}+\eta\,. \qquad \text{(15)}$$

The boundary conditions for $t>0$ are as follow :

at the upper end $x=0$, $H=H_0+h_0+F(t)$, ...(16)

where H_0 and h_0 are numerical constants and

$$\overline{F(t)}=0\,. \qquad \text{(17)}$$

at the lower end $x=x_1$,

either

$$\frac{\partial H}{\partial x}=0\,, \qquad \text{(18)}$$

which is the case where x_1 is very large or the river empties into a lake or a dam,

or

$$H(x_1,\ t)=H_0+H(t)\,, \qquad \text{(19)}$$

which is the case where at the lower end the river stage is regulated artificially or the river enters into the tidal estuary. As to the initial condition, we assume simply

$$H(x,\ 0)=H_0\,, \quad \text{..............................} (20)$$

i. e., before the flood the river has a uniform depth.

As the equation (14) is non-linear, we must content ourselves with approximation. In order to get an approximate solution, we assume the solution is expressible by the functional series[1]

$$H=(H_0+h_0)\left\{1+\frac{\varphi_1}{H_0+h_0}+\frac{\varphi_2}{(H_0+h_0)^2}+\dots\right\}, \quad \text{......} (21)$$

where φ_1, φ_2, ... satisfy the condition

$$\text{at} \quad x=0, \quad \varphi_1=F, \quad \varphi_2=\varphi_3=\varphi_4=\cdots\cdots=0$$

Putting (21) into (14) and collecting the terms of the first order with respect to $1/(H_0+h_0)$ we get for the first approximate solution an equation

$$\frac{\partial\varphi_1}{\partial t}+\frac{3U_0}{2}\frac{\partial\varphi_1}{\partial x}=\mu_0\frac{\partial^2\varphi_1}{\partial x^2}, \quad \text{.....................} (22)$$

where

$$\mu_0=\frac{(H_0+h_0)U_0}{2i}+\eta\,, \quad \text{.....................} (23)$$

and

$$U_0=c\sqrt{(H_0+h_0)i}\,, \quad \text{........................} (24)$$

with the boundary conditions for $t>0$:

$$\text{at} \quad x=0, \qquad \varphi_1(0,\ t)=F(t), \quad \text{........................} (25)$$

$$\text{at} \quad x=x_1, \qquad \frac{\partial\varphi_1}{\partial x}=0\,, \quad \text{..................................} (26)$$

$$\text{or} \qquad \varphi_1(x_1,\ t)=H(t)-h_0, \quad \text{........................} (27)$$

and the initial condition:

$$\text{at} \quad t=0, \qquad \varphi_1(x,\ 0)=-h_0. \quad \text{...........................} (28)$$

In the same manner, for the second approximate solution φ_2 we get an equation

$$\frac{\partial\varphi_2}{\partial t}+\frac{3U_0}{2}\frac{\partial\varphi_2}{\partial x}=\mu_0\frac{\partial^2\varphi_2}{\partial x^2}+\Gamma(x,\ t), \quad \text{..............} (29)$$

where

$$\Gamma(x,\ t)=\frac{3U_0}{4i}\left(\frac{\partial\varphi_1}{\partial x}\right)^2-\frac{3U_0}{4H_0}\varphi_1\left(\frac{\partial\varphi_1}{\partial x}\right), \quad \text{..............} (30)$$

with the boundary conditions for $t>0$:

$$\text{at} \quad x=0, \qquad \varphi_2(0,\ t)=0\,, \qquad \text{(31)}$$

$$\text{at} \quad x=x_1, \qquad \frac{\partial \varphi_2}{\partial x}=0\,, \qquad \text{(32)}$$

or

$$\varphi_2(x_1,\ t)=0\,, \qquad \text{(33)}$$

and the initial cóndition :

$$\text{at} \quad t=0, \qquad \varphi_2(x,\ 0)=0\,. \qquad \text{(34)}$$

(2) Solution of the diffferential equation

In the following we shall mainly concern with the first approximation. The first approximation is given by

$$H=H_0+h_0+\varphi_1(x,\ t)\,, \qquad \text{(35)}$$

where φ_1 is the solution of the eq. (22). The coefficient μ_0 is composed of two terms. The value of the one may be greater than the other, but they will probably be of the same order, so that the either of the two will not be negligible.

So far as the form of the diffusion coefficient η is not known, it will be wise to treat the coefficient μ_0 as a certain numerical constant to be determined by observation. This procedure is frequently used in meteorology and oceanography giving results of sufficient approximation. In the following we shall simply denote μ for μ_0.

The solution φ_1 under the boundary conditions (25), (26) and initial condition (28) is then given by

$$\varphi_1=-h_0+e^{\frac{\omega}{2\mu}x}\sum_1^{\infty}A_n\left\{\left(\frac{\omega}{2\mu}\right)^2+\xi_n^2\right\}\mu\ \sin\ \xi_n x$$
$$\times\int_0^t\{F(\lambda)+h_0\}\exp\left\{-\left[\xi_n^2+\left(\frac{\omega}{2\mu}\right)^2\right]\mu(t-\lambda)\right\}d\lambda\,, \qquad \text{(36)}$$

where

$$\omega=\frac{3}{2}U_0\,, \qquad \text{(37)}$$

$$A_n=\frac{2\xi_n}{x_1\left\{\left(\frac{\omega}{2\mu}\right)^2+\xi_n^2\right\}+\frac{\omega}{2\mu}}\,, \qquad \text{(38)}$$

and ξ_n is the roots of the equation

$$\tan\ \xi_n x_1=-\frac{2\mu\xi_n}{\omega}\,. \qquad \text{(39)}$$

When x_1 tends to infinity, we have

$$\varphi_1 = -h_0 + \frac{x}{2\sqrt{\pi\mu}} e^{\frac{\omega}{2\mu}x} \int_0^t \{F(\lambda)+h_0\} \frac{\exp\left\{-\frac{\omega^2}{4\mu}(t-\lambda) - \frac{x^2}{4\mu(t-\lambda)}\right\}}{(t-\lambda)^{3/2}} d\lambda$$

$$= -h_0' + \frac{2}{\sqrt{\pi}} e^{\frac{\omega}{2\mu}x} \int_{\frac{x}{2\sqrt{\mu t}}}^{\infty} \left\{F\left(t-\frac{x^2}{4\mu\xi^2}\right)+h_0\right\} \exp\left\{-\xi^2 - \frac{\left(\frac{\omega}{2\mu}\right)^2 x^2}{4\xi^2}\right\} d\xi$$

$$\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

In the case

$$F(t)=0,$$

we get from the formulae (35), (40),

$$H = H_0 + \frac{2h_0}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\mu t}}}^{\infty} \exp\left\{\frac{\omega x}{2\mu} - \xi^2 - \frac{\left(\frac{\omega}{2\mu}\right)^2 x^2}{4\xi^2}\right\} d\xi$$

$$= H_0 + h_0 - \frac{2h_0}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\mu t}}} \exp\left\{\frac{\omega x}{2\mu} - \xi^2 - \frac{\left(\frac{\omega}{2\mu}\right)^2 x^2}{4\xi^2}\right\} d\xi, \ldots\ldots\ldots (41)$$

or

$$\frac{H-H_0}{h_0} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\mu t}}} \exp\left\{\frac{\omega x}{2\mu} - \xi^2 - \frac{\left(\frac{\omega}{2\mu}\right)^2 x^2}{4\xi^2}\right\} d\xi. \ldots\ldots (41)'$$

This is a very useful formula for the later discussions.

When the function $F(t)$ is made of the sum of harmonic terms such as

$$F_n(t) = H_n \sin \gamma_n t, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

then assuming

$$t \rightarrow \infty, \qquad x_1 \rightarrow \infty,$$

we get from (40) the solution of elementary flood waves as follows:

$$\varphi_{1n} = H_n \exp\left\{\left(\frac{\omega}{2\mu} - p_n\right)x\right\} \sin(\gamma_n t - q_n x), \ \ldots\ldots\ldots (43)$$

where

$$\begin{matrix} p_n \\ q_n \end{matrix} = \sqrt{\frac{\sqrt{\left(\frac{\omega^2}{4\mu}\right)^2 + \gamma_n^2} \pm \frac{\omega^2}{4\mu}}{2\mu}}. \ \ldots\ldots\ldots\ldots\ldots\ldots (44)$$

For the waves of long period such as

$$\frac{\omega^2}{4\mu} \gg \gamma_n ,$$

it will be easily seen from the expression (44),

$$q_n \simeq \frac{\gamma_n}{\omega} , \qquad p_n \simeq \frac{\omega}{2\mu} , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

so that

$$\varphi_{1n} = H_n \sin\left(\gamma_n t - \frac{\gamma_n}{\omega} x\right) , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

in other words, the flood wave propagates with the velocity ω and does not damp, which is the case treated in the classical theory of flood waves.

On the other hand, for the waves of short period such as

$$\frac{\omega^2}{4\mu} \ll \gamma_n ,$$

we get the relations

$$p_n \simeq q_n \simeq \sqrt{\frac{\gamma_n}{2\mu}} . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

In this case, the flood wave propagates with greater velocity than ω and damps quickly. These relations explain the reason why flood wave steepens at the foreside and flattens gradually as it proceeds downstream.

In this place we shall touch on the second approximation. Using the value of the first approximation φ_1, we can calculate the function $\Gamma(x, t)$. The solution of the second approximate function φ_2 under the conditions (31), (32) (34) and in the case $x_1 \rightarrow \infty$ is then given by

$$\varphi_2 = \frac{2}{\pi} \int_0^x \frac{\partial}{\partial x} \left\{ \exp\left[\frac{\omega}{2\mu}(x-\xi)\right] \right\} \int_0^\infty \frac{\alpha \sin\{\alpha(x-\xi)\}}{\alpha^2 + \left(\frac{\omega}{2\mu}\right)^2} d\alpha$$

$$\times \int_0^t \Gamma(\xi, \tau) \exp\left\{-\left[\alpha^2 + \left(\frac{\omega}{2\mu}\right)^2\right]\mu(t-\tau)\right\} d\tau \Big\} d\xi \quad \ldots\ldots\ldots\ldots (48)$$

Because of the character of the function $\Gamma(x, t)$, over harmonic and combined harmonic flood waves appear in the second approximation. But owing to their short periods, they damp away quickly. This is probably the main reason why the second and higher approximate functions φ_2, φ_3, ... do not sensibly affect the propagation of flood waves—a fact which will be shown in the next chapter.

(3) Effects of tributaries and distributaries on the flood waves

The effects of tributaries and branching rivers upon the flood waves in the main stream are one of problems of interest. They will be treated as the sources and sinks distributed along the main river. In this case, the equation of continuity is given by

$$\frac{\partial H}{\partial t}=-\frac{\partial Q}{\partial x}+\eta\frac{\partial^2 H}{\partial x^2}+S(x,\ t), \quad \ldots\ldots\ldots\ldots\ldots\ldots(49)$$

where $S(x,\ t)$ represents the amount of discharge into or out of the main stream per unit length along the channel and per unit time divided by the breadth of the channel, and the equation of the first approximate solution φ_1 takes the form

$$\frac{\partial \varphi_1}{\partial t}+\omega\frac{\partial \varphi_1}{\partial x}=\mu\frac{\partial^2 \varphi_1}{\partial x^2}+S(x,\ t), \quad \ldots\ldots\ldots\ldots\ldots\ldots(50)$$

with the same boundary conditions as before. As the eq. (50) is linear, the effect of the function $S(x,\ t)$ which we shall denote φ_1' will be simply additive to the solution φ_1 already mentioned. The equation of φ_1' is, therefore, given by

$$\frac{\partial \varphi_1'}{\partial t}+\omega\frac{\partial \varphi_1'}{\partial x}=\mu\frac{\partial^2 \varphi_1'}{\partial x^2}+S(x,\ t), \quad \ldots\ldots\ldots\ldots\ldots\ldots(50)'$$

with the boundary conditions for $t>0$:

$$\left.\begin{array}{lll} \text{at} & x=0, & \varphi_1'(0,\ t)=0 \\ \text{at} & x\to\infty, & \dfrac{\partial \varphi'}{\partial x}=0\,, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(51)$$

and the initial condition :

$$\text{at} \quad t=0, \quad \varphi_1'(x,\ 0)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(51)'$$

The form of the eq. (50)′ and the conditions (51), (51)′ are entirely same with those of φ_2, so that the effect φ_1' is given by the formula (48) where the function $S(x,\ t)$ now stands for the function $\Gamma(x,\ t)$.

3. Practical Examples

(1) Unit graph method

In order to calculate the first aporoximate solution it is necessary to know the form of the function $F(t)$. When the functional form is simple, it may be possible to evaluate the integral analytically, but in actual cases it is desirable to device some practical method suitable for numerical computation. For this purpose, we divide the time coordinate into many elementary parts of equal interval t_0 which is selected conveniently for respective cases and assume that the functian $F(t)$ is constant in each elementary interval. Then the solution φ_1 is expressed as the sum of elementary integrals calculated for each interval. The elementary integral is called the unit graph and the corresponding flood is called the unit flood. This method of evaluation is usually called the unit graph method and is proved to be very effective in the linear problem[2].

For a solitary unit flood, we have

$$\overline{H}=\overline{H}_0+\overline{h}_0+\overline{F(t)} \simeq H_0 \quad \text{for large } t, \quad \text{i. e.} \quad h_0 \simeq 0 .$$

So, if we take

$$\left.\begin{array}{ll} F=0, & t \leqslant 0 , \\ F=h, & 0 < t \leqslant t_0, \\ F=0, & t_0 < t , \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (52)$$

then h represents the height of the unit flood at the upper end above normal river stage. Since the problem is linear, the solution under the conditions (52) i. e. the unit graph, is equivalent to the sum of two solutions φ_1' and φ_1'' with respective conditions such that

$$\text{at} \quad x=0, \quad \left.\begin{array}{ll} F=0, & t \leqslant 0 , \\ F=h, & 0 < t , \end{array}\right\} \cdots\cdots\cdots\cdots\cdots (53)$$

and

$$\text{at} \quad x=0, \quad \left.\begin{array}{ll} F=0, & t \leqslant t_0 , \\ F=-h, & t_0 < t . \end{array}\right\} \cdots\cdots\cdots\cdots\cdots (54)$$

The former solution is given by the formula (41) where h now stands for h_0, and the latter is also given by the same formula where, however, the time origin is displaced by t_0 and $-h$ stands for h_0. The solution of unit graph is, therefore, reduced to the evaluation of (41) and this is an easy task. A numerical example of the solitary unit flood is shown in Fig. 1–A where the ratio of flood height $(H-H_0)/h$ is plotted against time for various distances. This example is based on the following data:

$$\omega=70 \text{ cm/sec}, \qquad \mu=10^7 \text{ c. g. s.}, \qquad t_0=5 \text{ hours},$$

$$x=2.2, \ 14, \ 21 \text{ and } 32 \text{ km}.$$

The form of an unit flood flattens gradually as it propagates downstream. It is asymmetric, the slope of the foreside being steeper than the slope of the backside.

When two unit floods occur successively, they merge gradually into a single flood as they propagate downstream. An example is shown in Fig. 1–B where both unit floods are assumed to be same as that in Fig. 1–A and the interval of the floods is 2.5 hours.

In the practical application of the unit graph method to the flood of any form, it will be a matter of concern how to choose the time interval of the unit flood. To see the degree of approximation, an example will be shown ; We assume, for the example, at the upper end

$$F(t)=\sin \gamma t, \qquad -\infty < t < +\infty, \quad \text{..................} (55)$$

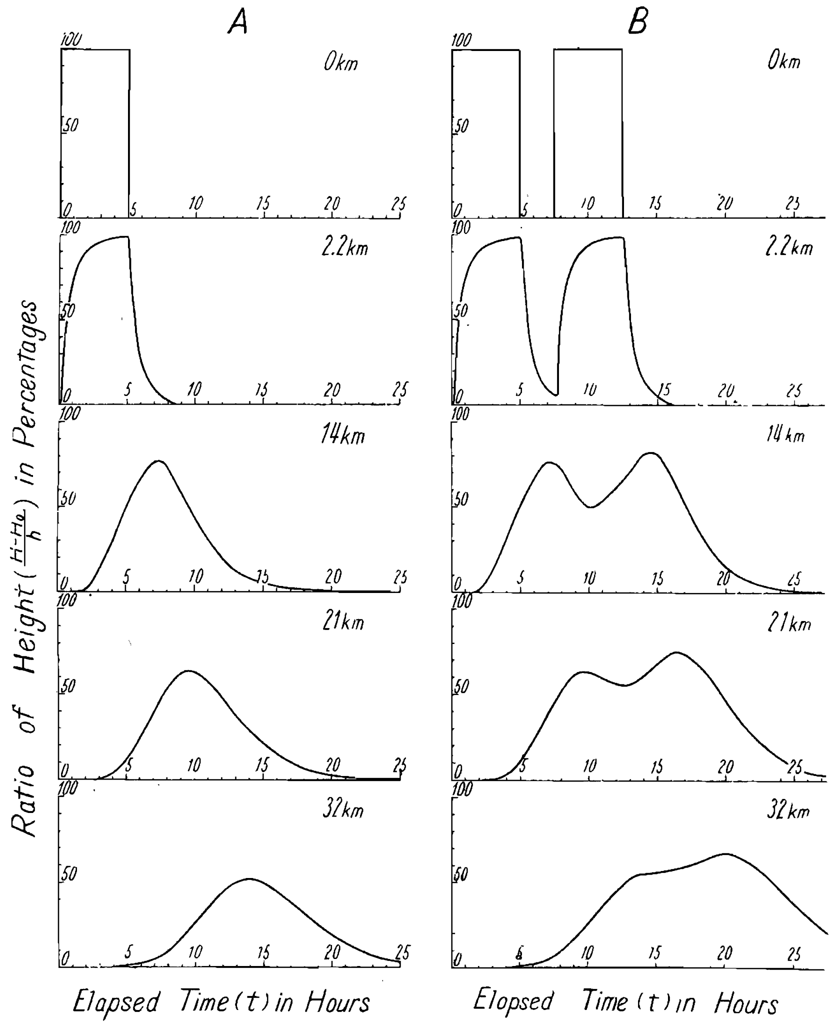

Fig. 1–A. Propagation of a solitary unit flood.
Fig. 1–B. Propagation of two successive unit floods.

then the flood wave at any point x is given by the expression (43).

Putting

$$\omega=70 \text{ cm/sec}, \quad \mu=10^7 \text{ c. g. s.}, \quad \frac{2\pi}{\gamma}=8 \text{ hours},$$

we get at the station $x=14$ km

$$\varphi_1=0.32 \sin\{\gamma(t-4.5)\}, \quad t \text{ in hrs.} \quad \text{.................} (56)$$

The graphs of (55) and (56) are shown in Fig. 2 in full lines. On the other hand, we take, in trial,

$$t_0=1 \text{ hour},$$

and take the mean value of sin γt for each interval as shown in the same Figure. Assuming the same value of ω we calculate the corresponding unit graphs successively at the same station by the procedure mentioned above. Then summing up these graphs, we get the flood wave at the station which is shown in the Figure as series of dots. The agreement is fairly good in spite of such a rough substitution.

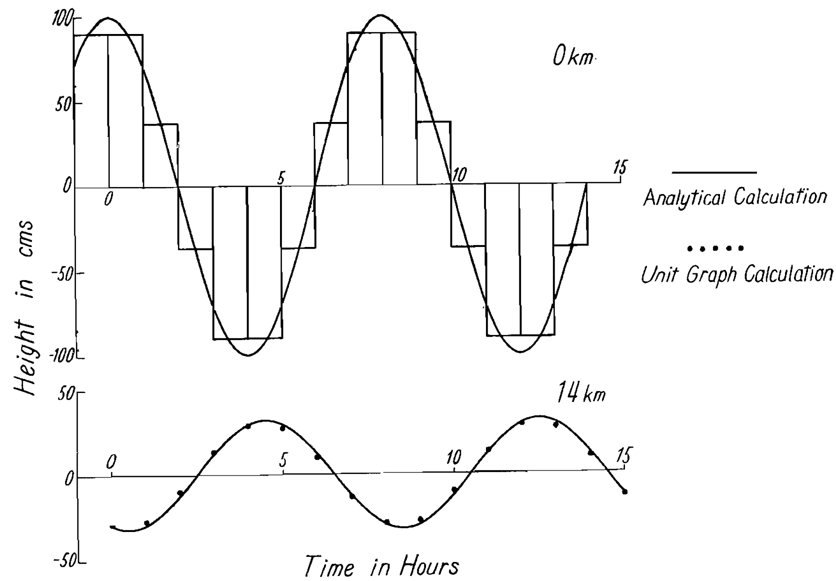

Fig. 2. Comparison of analytical method with unit graph calculation.

(2) Comparison with observations

The merit of a physical theory is estimated by the degree of agreement between theoretical consequences and observational evidences. As the flood of any form is composed of a number of unit floods, it is best to compare the theory with observation on a single unit flood. In 1943 Hideo Kikkawa made some observations on an unit flood in the Yedo River(3). The Yedo River is one of branch rivers of the Tone River, one of the largest rivers of Japan. It branches from the main river near Sakai and taking the south south easterly course of about 60 km pours into the Bay of Tokyo. Near the head of the Yedo River, a lock is build for the regulation of its discharge. Operating the lock, Kikkawa produced an unit flood artificially and pursued it downstream as far as 32 kilometres. Some of his results is shown in Table 1.

Table 1. Observational results of solitary unit flood in the Yedo River.

Date of observation. Dec. 7, 1943				
Normal river depth H_0 ca. 60cm	Height of flood at the lock h 90cm	Slope of the bed i ca. 2×10^{-4}	Duration of flood at the lock t_0 5 hours	
Distance below the lock x in km	Height of the crest avove normal river stage in cm	Duration of flood in hours	Arrival time of the crest in hours	Arrival time of the front in hours
0	90	5	2.5	0
2.2	87	8	4.5	0.9
14	68	11	8.0	3
21	58	12.5	10.1	4.5
32	49	15	12.0	7.6

From these materials we assume

$$\omega=70\ \text{cm/sec},\quad \mu=10^7\ \text{c. g. s.},\quad t_0=5\ \text{hours},$$

then the corresponding unit graphs at $x=2.2$, 14, 21 and 32 km are entirely same to those already shown in Fig. 1–A. From these graphs the heights of the flood crest above normal stage, durations of the flood and the arrival times of the crest and front were estimated. A comparison of these estimated values with those in Tab. 1 is shown in Fig. 3. In the estimation of the duration and arrival time of the front (by the term front is meant the foremost part of the flood), some ambiguities may be expected as to the stage of the river at which the determination is made. For the theoretical values in Fig. 3 two stages were asssumed such as

$$\frac{H-H_0}{h}=5\%\quad\text{and}\quad 10\%.$$

As the Figure shows, the observed points mainly lie between them. Generally speaking the agreement is said to be excellent.

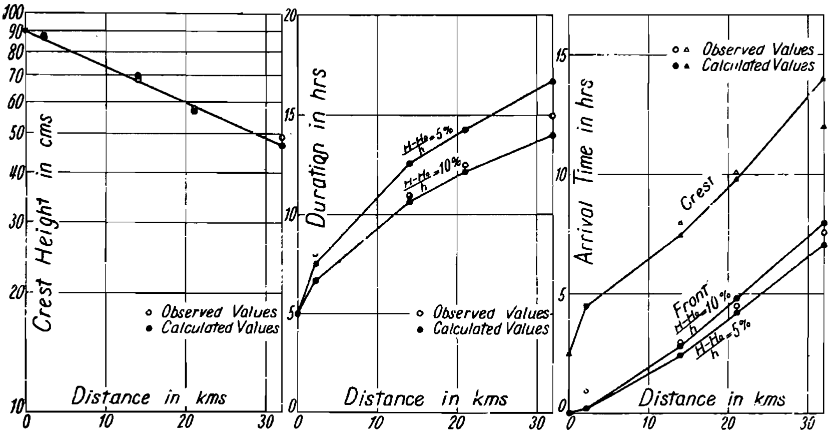

Fig. 3. Comparison of observational results with theoretical computation (solitary unit flood in the Yedo River).

(3) More general cases

In the discussions so far developed the depth and the breadth of a river were assumed to be uniform in the mean value. Although this assumption is very practical, there are many cases where in the mean they must be rather regarded as some functions of x. In these cases the equation of flood waves assumes somewhat a complicated form. Since the physical nature of the propagation of flood waves does not change, it will be surely inferred that, in these cases also, the first approximation which of course of a linear character, gives a sufficient approximation and flood of any form will be composed of a number of unit floods. The analytical solution of the unit graph is very troublesome to obtain and even if obtained, it will be of the form not suitable for the numerical computations. But, if we get by an observation the hydrograph of any flood at some downstream point together with the corresponding boundary condition at the upper end, we can obtain the unit graph at the point numerically; We divide the time into constant intervals t_0 and assume the function $F(t)$ given by the hydrograph at the upper end as such

$$\left.\begin{array}{lllll} \text{for} & 0 < t \leqslant t_0, & t_0 < t \leqslant 2t_0, & 2t_0 < t \leqslant 3t_0, & \cdots\cdots\cdots \\ F(t) = & F_1, & F_2, & F_3\ , & \cdots\cdots\cdots \end{array}\right\} \cdots (57)$$

where F_1, F_2, F_3, ..., are numerical constants. Let the unit graph at any point $x=x_0$ under the condition

$$\left.\begin{aligned} F&=0, & t&\leqslant 0, \\ F&=1, & 0<t&\leqslant t_0, \\ F&=0, & t_0&<t, \end{aligned}\right\} \qquad (58)$$

be denoted by $\phi(t)$. By the assumption of the unit graph

$$\phi(t)=0, \qquad t\leqslant 0. \qquad (59)$$

Then the flood wave at the point x_0 will be given by

$$H=H_0+F_1\phi(t)+F_2\phi(t-t_0)+F_3\phi(t-2t_0)+\cdots\cdots \qquad (60)$$

As the height of the flood $H-H_0$ is given by the hydrograph, its value is known for every value of t, so we get many algebraic equations containing the function $\phi(t)$ as unknown. Solving these equations successively we can construct the function $\phi(t)$. Once $\phi(t)$ is known, the flood of any form will be predicted at the point x_0 by the hydrograph $F(t)$ at the upper end. The writer is now preparing to obtain the flood characteristics of the main rivers of Japan from this point of view.

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